JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 49, 535–544 (1975)

On the Approximate Solution of the Singular Problem of Cauchy for Ordinary Differential Equations

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Submitted by G. Dahlquist

In the present paper an approximate solution of the singular problem of Cauchy for the ordinary differential equation of mth order is constructed and, by the method of finite differences, sufficient conditions are found for the convergence to the exact solution when the mesh width tends to zero.

1. FORMULATION OF THE MAIN RESULT

Consider the Cauchy problem

$$u^{(m)} = f(t, u, u', \dots, u^{(m-1)}), \tag{1.1}$$

$$\lim_{t\to 0} u^{(i)}(t) = 0 \qquad (i = 0, \dots, m-1), \tag{1.2}$$

where $f(t, x_0, \dots, x_{m-1})$ is a continuous function, given in the domain

$$\mathscr{D} = (0, 1] \times R^m$$

and R^m is the m-dimensional Euclidean space. Besides, $f(t, x_0, \dots, x_{m-1})$ is not, in general, integrable with respect to t on the segment $[0, 1]$, having a singularity when $t = 0$. In this sense the problem (1.1), (1.2) is singular.

The solution of this problem will be sought in the class of functions which are m times continuously differentiable in the interval $[0, 1]$.

Criteria for the existence and uniqueness of the solution of the singular problem (1.1), (1.2) are contained in [3, 4, 6]. As regards the question of its approximate solution, it is not yet sufficiently investigated.

Problem (1.1), (1.2) corresponds to the following discrete problem of Cauchy

$$\Delta^m u_i = \frac{1}{n^m} f\left(\frac{i+1}{n}, u_i, n\Delta u_i, \dots, n^{m-1}\Delta^{m-1}u_i\right) \qquad (i = 0, \dots, n-1-m), \tag{1.3}$$

$$\Delta^k u_0 = 0 \qquad (k = 0, \dots, m-1), \tag{1.4}$$

where

$$n \geqslant m - 1,$$

$$\Delta^0 u_i = u_i \,, \qquad \Delta^1 u_i = \Delta u_i = u_{i+1} - u_i \,, \qquad \Delta^k u_i = \Delta(\Delta^{k-1} u_i).$$

Below we prove the following:

THEOREM 1.1. *Let the function $f(t, x_0 ,..., x_{m-1})$ be continuous as in the domain $\mathscr{D}$ and satisfy the inequalities*

$$| f(t, 0,..., 0)| \leqslant a(t),$$

$$| f(t, x_0 ,..., x_{m-1}) - f(t, y_0 ,..., y_{m-1})| \leqslant a(t) \sum_{k=0}^{m-1} t^{k+1-m} \, | x_k - y_k \,| \,, \tag{1.5}$$

where the function $a(t) \in L(0, 1)$ is continuous and nonincreasing in the interval $0 < t \leqslant 1$. Then

(i) *The problem* (1.1), (1.2) *has a unique solution $u(t)$;*

(ii) *For any $n \geqslant m + 1$ the problem* (1.3), (1.4) *has a unique solution $u_0^n ,..., u_{n-1}^n$ and*

(iii) *The following condition*

$$\lim_{n\to+\infty} \left| u^{(k)} \left(\frac{i+1}{n}\right) - n^k \Delta^k u_i^n \right| = 0 \qquad (k = 0,..., m-1) \tag{1.6}$$

is satisfied uniformly with respect to i.

2. VARIOUS LEMMAS

LEMMA 2.1.[1] *Let $a_i \geqslant 0$, $0 \leqslant c_i \leqslant c_{i+1}$ $(i = 0,..., n-1)$ and*

$$v_0 \leqslant c_0 \,, \qquad v_i \leqslant c_i + \sum_{k=0}^{i-1} a_k v_k \qquad (i = 1,..., n). \tag{2.1}$$

Then

$$v_i \leqslant c_i \exp\left(\sum_{k=0}^{i-1} a_k\right) \qquad (i = 1,..., n). \tag{2.2}$$

Proof. We may consider, without loss of generality, that $v_j \geqslant 0$ $(j = 0,..., n)$.

[1] This lemma is the discrete analog of the well-known lemma of R. Bellman [2, p. 188]. Some other analogs of R. Bellman's lemma ([1, p. 72] and [5, p. 311]) are also known.

Let $i \in \{1,..., n\}$ and $\epsilon > 0$. Suppose

$$w_0 = c_i + \epsilon, \qquad w_j = c_i + \epsilon + \sum_{k=0}^{j-1} a_k v_k \qquad (j = 1,..., i).$$

Then in view of (2.1),

$$v_j < w_j \qquad (j = 0,..., i - 1).$$

Therefore,

$$w_{j+1} - w_j = a_j v_j \leqslant a_j w_j \qquad (j = 0,..., i - 1). \tag{2.3}$$

Since $w_{j+1} \geqslant w_j > 0$ $(j = 0,..., i - 1)$, it is clear that

$$\frac{w_{j+1}}{w_j} - 1 \geqslant \ln \frac{w_{j+1}}{w_j} \qquad (j = 0,..., i - 1),$$

and according to this, from (2.3) we have

$$\ln \frac{w_{j+1}}{w_j} \leqslant a_j \qquad (j = 0,..., i - 1).$$

By summing these inequalities, we find

$$\ln \frac{w_i}{c_i + \epsilon} \leqslant \sum_{j=0}^{i-1} a_j .$$

Consequently,

$$v_i \leqslant w_i \leqslant (c_i + \epsilon) \exp \left(\sum_{j=0}^{i-1} a_j \right) .$$

From this, due to the arbitrariness of i and ϵ, inequalities (2.2) follow imediately. The lemma is thus proved.

LEMMA 2.2. *Let* $n \geqslant m + 1$

$$| \Delta^m u_i | \leqslant B_i + \sum_{k=0}^{m-1} A_{ik} | \Delta^k u_i | \qquad (i = 0,..., n - 1 - m) \tag{2.4}$$

and conditions (1.4) *be fulfilled, where* A_{ik} *and* B_i $(i = 0,..., n - 1 - m;$ $k = 0,..., m - 1)$ *are nonnegative constants. Then*

$$u_i = 0 \qquad (i = 0,..., m - 1) \tag{2.5}$$

and

$$(i+1)^{k-m+1} \mid \Delta^k u_i \mid \leqslant \left(\sum_{j=0}^{i+k-m} B_j \right) \exp \left[\sum_{s=m-1}^{i+k-1} \sum_{j=0}^{m-1} (s+1-j)^{m-1-j} A_{s-jj} \right]$$
$$(i = m-k,..., n-1-k; k = 0,..., m-1). \qquad (2.6)$$

Proof. It is easily verified that

$$\Delta^k u_i = \sum_{j=k}^{i+k} \binom{i}{j-k} \Delta^j u_0 \quad (i = 0,..., m-1-k; k = 0,..., m-1), \qquad (2.7)$$

and (see [7, pp. 327–329]),

$$\Delta^k u_i = \sum_{j=k}^{m-1} \binom{i}{j-k} \Delta^j u_0 + \sum_{j=0}^{i+k-m} \binom{i-j-1}{m-k-1} \Delta^m u_j$$
$$(i = m-k,..., n-1-k; k = 0,..., m-1) \qquad (2.8)$$

where

$$\binom{s}{0} = 1 \ (s = 0, 1, ...),$$
$$\binom{s}{k} = \frac{s(s-1) \cdots (s-k+1)}{k!} \qquad (k = 1, 2,...; s = 0, 1,...).$$

According to (1.4), from (2.7) we immediately obtain the validity of equalities (2.5).

Due to (1.4) and (2.4), from (2.8) we get

$$\mid \Delta^k u_{i-k} \mid \leqslant \sum_{j=0}^{i-m} \binom{i-k-j-1}{m-k-1} \left(B_j + \sum_{s=0}^{m-1} A_{js} \mid \Delta^s u_j \mid \right).$$

Since

$$\binom{i-k-j-1}{m-k-1} < (i-k+1)^{m-k-1} \qquad (i = m,..., n-1; k = 0,..., m-1),$$

from the latter inequalities we have

$$(i+1-k)^{k-m+1} \mid \Delta^k u_{i-k} \mid \leqslant \sum_{j=0}^{i-m} B_j + \sum_{s=0}^{i-m} \sum_{j=0}^{m-1} A_{sj} \mid \Delta^j u_s \mid$$
$$(i = m,..., n-1; k = 0,..., m-1). \qquad (2.9)$$

Taking into account (2.5), it can be easily concluded that

$$\sum_{s=0}^{i-m} \sum_{j=0}^{m-1} A_{sj} \mid \Delta^j u_s \mid \leqslant \sum_{s=0}^{m-2} \sum_{j=0}^{s} A_{s-jj} \mid \Delta^j u_{s-j} \mid + \sum_{s=m-1}^{i-1} \sum_{j=0}^{m-1} A_{s-jj} \mid \Delta^j u_{s-j} \mid$$
$$\leqslant \sum_{s=m-1}^{i-1} \sum_{j=0}^{m-1} A_{s-jj} \mid \Delta^j u_{s-j} \mid$$
$$(i = m,..., n-1; k = 0,..., m-1). \qquad (2.10)$$

Introduce the notations

$$v_i = \max\{(i+1-k)^{k-m+1} \mid \Delta^k u_{i-k} \mid : k = 0,\dots, m-1\} \qquad (i = m-1, m,\dots).$$

Then from (2.5), (2.9) and (2.10) we have

$$v_{m-1} = 0,$$

$$v_i \leqslant \sum_{j=0}^{i-m} B_j + \sum_{s=m-1}^{i-1} \left[\sum_{j=0}^{m-1} (s+1-j)^{m-1-j} A_{s-jj} \right] v_s \qquad (i = m,\dots, n-1).$$

Hence, due to Lemma 2.1, we obtain

$$v_i \leqslant \left(\sum_{j=0}^{i-m} B_j \right) \exp \left[\sum_{s=m-1}^{i-1} \sum_{j=0}^{m-1} (s+1-j)^{m-1-j} A_{s-jj} \right] \qquad (i = m,\dots, n-1).$$

In this way the validity of inequalities (2.6) is proved.

3. Proof of Theorem 1.1

As it has been proved in [3] by the conditions of Theorem 1.1, the problem (1.1), (1.2) has one and only one solution $u(t)$. On the other hand, it is evident that for any $n > m+1$ the Problem (1.3), (1.4) has one and only one solution $u_0{}^n,\dots, u_{n-1}^n$. Thus our aim is to prove that the condition (1.6) is satisfied uniformly with respect to i.

It is easily seen that

$$| u^{(k)}(t)| \leqslant \eta(t)\, t^{m-1-k} \quad \text{when} \quad 0 \leqslant t \leqslant 1 \quad (k = 0,\dots, m-1) \tag{3.1}$$

and

$$| u^{(k)}(t) - u^{(k)}(\tau)| \leqslant \eta(| t - \tau |) \quad \text{when} \quad 0 \leqslant \tau, t \leqslant 1 \quad (k = 0,\dots, m-1), \tag{3.2}$$

where $\eta(t)$ is a function which is continuous and nondecreasing and satisfies the condition

$$\eta(0) = 0.$$

Assume

$$v_i^n = u\left(\frac{i+1}{n}\right) - \sum_{j=0}^{m-1} \binom{i}{j} \Delta^j u \left(\frac{1}{n}\right) \qquad (i = 0,\dots, n-1),$$

then, by differencing with respect to i,

$$\Delta^k v_i^n = \Delta^k u\left(\frac{i+1}{n}\right) - \sum_{j=k}^{m-1} \binom{i}{j-k} \Delta^j u\left(\frac{1}{n}\right)$$

$$(i = 0,\ldots, n-1-k;\ k = 0,\ldots, m-1) \qquad (3.3)$$

and

$$\Delta^m v_i^n = \Delta^m u\left(\frac{i+1}{n}\right) \qquad (i = 0,\ldots, n-1-m). \qquad (3.4)$$

We can evidently find the points

$$\xi_{ik}^n \in \left[\frac{i+1}{n}, \frac{i+1+k}{n}\right] \qquad (i = 0,\ldots, n-1-k;\ k = 0,\ldots, m) \qquad (3.5)$$

such that

$$\Delta^k u\left(\frac{i+1}{n}\right) = \frac{1}{n^k} u^{(k)}(\xi_{ik}^n) \qquad (i = 0,\ldots, n-1-k;\ k = 0,\ldots, m). \qquad (3.6)$$

Due to (3.3) and (3.6)

$$u^{(k)}(\xi_{ik}^n) - n^k \Delta^k v_i^n = n^k \left[\Delta^k u\left(\frac{i+1}{n}\right) - \Delta^k v_i^n\right] = n^k \sum_{j=k}^{m-1} \binom{i}{j-k} \Delta^j u\left(\frac{1}{n}\right)$$

$$= \sum_{j=k}^{m-1} \binom{i}{j-k} n^{k-j} u^{(j)}(\xi_{0j}^n).$$

From these equalities, according to (3.1) and (3.5), we have

$$| u^{(k)}(\xi_{ik}^n) - n^k \Delta^k v_i^n | \leqslant \sum_{j=k}^{m-1} \binom{i}{j-k} n^{k-j} \eta(\xi_{0j}^n)\,(\xi_{0j}^n)^{m-1-j}$$

$$\leqslant \eta\left(\frac{m}{n}\right) \sum_{j=k}^{m-1} \binom{i}{j-k} n^{k-j} \left(\frac{j+1}{n}\right)^{m-1-j}$$

$$\leqslant m^{m-1} \eta\left(\frac{m}{n}\right) n^{k-m+1} \sum_{j=k}^{m-1} \binom{i}{j-k}$$

$$\leqslant m^m \eta\left(\frac{m}{n}\right) n^{k-m+1} (i+1)^{m-1-k}$$

$$(i = 0,\ldots, n-1-k;\ k = 0,\ldots, m-1).$$

Consequently,

$$| u^{(k)}(\xi_{ik}^n) - n^k \Delta^k v_i^n | \leqslant m^m \eta \left(\frac{m}{n}\right) \left(\frac{i+1}{n}\right)^{m-1-k}$$
$$(i = 0, \ldots, n-1-k;\ k = 0, \ldots, m-1). \qquad (3.7)$$

Hence, due to (3.2) and (3.5) it is clear that

$$\begin{aligned} &| u^{(k)}(\xi_{im}^n) - n^k \Delta^k v_i^n | \\ &\quad \leqslant | u^{(k)}(\xi_{im}^n) - u^{(k)}(\xi_{ik}^n)| + | u^{(k)}(\xi_{ik}^n) - n^k \Delta^k v_i^n | \qquad (3.8) \\ &\quad \leqslant (1 + m^m)\, \eta \left(\frac{m}{n}\right) \qquad (i = 0, \ldots, n-1-m;\ k = 0, \ldots, m-1). \end{aligned}$$

By virtue of (3.1) and (3.8)

$$\begin{aligned} &n^k | \Delta^k v_i^n | + | u^{(k)}(\xi_{im}^n)| \\ &\quad \leqslant | n^k \Delta^k v_i^n - u^{(k)}(\xi_{im}^n)| + 2\, | u^{(k)}(\xi_{im}^n)| \\ &\quad \leqslant (1 + m^m)\, \eta \left(\frac{m}{n}\right) \left(\frac{i+1}{n}\right)^{m-1-k} + 2\eta \left(\frac{i+1+m}{n}\right) \left(\frac{i+1+m}{n}\right)^{m-1-k} \\ &\quad \leqslant c_0 \left(\frac{i+1+m}{n}\right)^{m-1-k} \qquad (i = 0, \ldots, n-1-m;\ k = 0, \ldots, m-1), \end{aligned} \qquad (3.9)$$

where

$$c_0 = (3 + m^m)\, \eta(1).$$

By (3.4) and (3.6)

$$\Delta^m v_i^n = \frac{1}{n^m} f\left(\frac{i+1}{n}, v_i^n, n\Delta v_i^n, \ldots, n^{m-1}\Delta^{m-1} v_i^n\right) + \frac{1}{n^m} r_i^n$$
$$(i = 0, \ldots, n-1-m), \qquad (3.10)$$

where

$$\begin{aligned} r_i^n &= f(\xi_{im}^n, u(\xi_{im}^n), u'(\xi_{im}^n), \ldots, u^{(m-1)}(\xi_{im}^n)) \\ &\quad - f\left(\frac{i+1}{n}, v_i^n, n\Delta v_i^n, \ldots, n^{m-1}\Delta^{m-1} v_i^n\right) \qquad (i = 0, \ldots, n-1-m). \end{aligned} \qquad (3.11)$$

From (1.5) it is clear that

$$\begin{aligned} | f(t, x_0, \ldots, x_{m-1})| &\leqslant | f(t, 0, \ldots, 0)| + | f(t, x_0, \ldots, x_{m-1}) - f(t, 0, \ldots, 0)| \\ &\leqslant a(t) \left[1 + \sum_{k=0}^{m-1} t^{k+1-m}\, | x_k - y_k |\right]. \end{aligned}$$

On account of this inequality from (3.5), (3.9) and (3.11) we obtain

$$| r_i^n | \leqslant a(\xi_{im}^n)\left[1 + \sum_{k=0}^{m-1} (\xi_{im}^n)^{k+1-m} \, | u^{(k)}(\xi_{im}^n)|\right]$$

$$+ a\left(\frac{i+1}{n}\right)\left[1 + \sum_{k=0}^{m-1} \left(\frac{i+1}{n}\right)^{k+1-m} | n^k \Delta^k v_i^n |\right]$$

$$\leqslant a\left(\frac{i+1}{n}\right)\left[2 + \sum_{k=0}^{m-1} \left(\frac{i+1}{n}\right)^{k+1-m} (| u^{(k)}(\xi_{im}^n)| + | n^k \Delta^k v_i^n |)\right]$$

$$\leqslant a\left(\frac{i+1}{n}\right)\left[2 + c_0 \sum_{k=0}^{m-1} \left(\frac{i+1+m}{i+1}\right)^{m-1-k}\right]$$

$$\leqslant c_1 a\left(\frac{i+1}{n}\right) \qquad (i = 0,..., n-1-m), \tag{3.12}$$

where

$$c_1 = 2 + c_0(m+1)^m.$$

Let

$$w_i^n = u_i^n - v_i^n \qquad (i = 0,..., n-1). \tag{3.13}$$

Then by virtue of (1.3)–(1.5), (3.3) and (3.10)

$$\Delta^k w_0^n = 0 \qquad (k = 0,..., m-1)$$

and

$$| \Delta^m w_i^n | \leqslant \frac{1}{n} a\left(\frac{i+1}{n}\right) \sum_{k=0}^{m-1} (i+1)^{k+1-m} \, | \Delta^k w_i^n | + \frac{1}{n^m} | r_i^n |$$

$$(i = 0,..., n-1-m)$$

Hence, according to Lemma 2.2, it follows that

$$\Delta^k w_i^n = 0 \;\; (i = 0,..., m-1-k; k = 0,..., m-1) \tag{3.14}$$

and

$$(i+1)^{k-m+1} \, | \Delta^k w_i^n | \leqslant \frac{1}{n^m}\left(\sum_{j=0}^{i+k-m} | r_j^n |\right) \exp\left[\frac{1}{n} \sum_{s=m-1}^{i+k-1} \sum_{j=0}^{m-1} a\left(\frac{s-j+1}{n}\right)\right]$$

$$(i = m-k,..., n-1-k; k = 0,..., m-1). \tag{3.15}$$

Since $a(t)$ is nonincreasing, we have

$$\frac{1}{n}\sum_{s=m-1}^{i+k-1}\sum_{j=0}^{m-1} a\left(\frac{s-j+1}{n}\right) \leqslant \frac{m}{n}\sum_{s=m-1}^{n-2} a\left(\frac{s-m+2}{n}\right) \leqslant m\int_0^1 a(t)\,dt$$

$$(i = m-k,..., n-1-k;\ k = 0,..., m-1).$$

Therefore, from (3.14), and (3.15) we have

$$n^k \mid \Delta^k w_i^n \mid \leqslant \frac{c_1}{n}\sum_{j=0}^{n-1-m} \mid r_j^n \mid \qquad (i = 0,..., n-1-k;\ k = 0,..., m-1), \tag{3.16}$$

where

$$c_2 = \exp\left[m\int_0^1 a(t)\,dt\right].$$

Due to (3.2), (3.5), (3.7), (3.13) and (3.16) we have

$$\left| u^{(k)}\left(\frac{i+1}{n}\right) - n^k\Delta^k v_i^n \right|$$
$$\leqslant n^k \mid \Delta^k w_i^n \mid + \mid u^{(k)}(\xi_{ik}^n) - n^k\Delta^k v_i^n \mid + \mid u^{(k)}\left(\frac{i+1}{n}\right) - u^{(k)}(\xi_{ik}^n)|$$
$$\leqslant (1+m^m)\,\eta\left(\frac{m}{n}\right) + \frac{c_2}{n}\sum_{j=0}^{n-1-m} \mid r_j^n \mid$$
$$(i = 0,..., n-1-k;\ k = 0,..., m-1).$$

Thus in order to prove the theorem it is sufficient to show that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1-m} \mid r_j^n \mid = 0. \tag{3.17}$$

Let us assume that for any $\delta \in (0, 1)$ and $\lambda \in (0, 1)$,

$$\omega_\lambda(\delta) = \max\{|f(t, x_0,..., x_{m-1}) - f(s, y_0,..., y_{m-1})|: \lambda \leqslant t, s \leqslant 1, \mid t-s \mid \leqslant \delta,$$
$$\mid x_i \mid + \mid y_i \mid \leqslant c_0, \mid x_i - y_i \mid \leqslant (1+m^m)\,\eta(\delta)\ (i = 0,..., m-1)\}.$$

It is clear that

$$\lim_{\delta\to 0} \omega_\lambda(\delta) = 0 \qquad \text{when} \qquad \lambda \in (0, 1). \tag{3.18}$$

On the other hand, according to (3.5), (3.8) and (3.9), from (3.11) we have

$$\mid r_i^n \mid \leqslant \omega_\lambda\left(\frac{m}{n}\right) \qquad \text{when} \qquad i \geqslant n\lambda - 1,\ \lambda \in (0, 1). \tag{3.19}$$

Let $\lambda \in (0, 1)$ and let $[n\lambda]$ be the integer part of $n\lambda$, then according to (3.12) and (3.19) we have

$$\begin{aligned}\frac{1}{n}\sum_{j=0}^{n-1-m} |r_j^n| &= \frac{1}{n}\sum_{j=0}^{[n\lambda]-1} |r_j^n| + \frac{1}{n}\sum_{j=[n\lambda]}^{n-1-m} |r_j^n| \\ &\leqslant c_1 \sum_{j=0}^{[n\lambda]-1} \frac{1}{n}\, a\left(\frac{j+1}{n}\right) + \frac{1}{n}\sum_{j=[n\lambda]}^{n-1-m} \omega_\lambda\left(\frac{m}{n}\right) \\ &\leqslant c_1 \int_0^\lambda a(t)\,dt + \omega_\lambda\left(\frac{m}{n}\right)\end{aligned}$$

when

$$n > \max\left\{\frac{m+1}{1-\lambda}, \frac{2}{\lambda}\right\}.$$

Hence, due to (3.18), we have

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1-m} |r_j^n| \leqslant c_1 \int_0^\lambda a(t)\,dt$$

when $\lambda \in (0, 1)$.

From this latter inequality the validity of (3.16) immediately follows. Thus the theorem is proved.

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